VETOR

Evaluation of the Convergence Interval of the Fixed Talbot Algorithm for Numerical Inversion of the Laplace Transform Applied to Solving the Wave Equation☆

Avaliação do intervalo de Convergência do Algoritmo Talbot Fixo de Inversão Numérica da Transformada de Laplace Aplicado na Resolução da Equação de Onda

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Abstract

This work examines the numerical inversion of the Laplace Transform via Fixed Talbot in an initial-boundary value problem for the wave equation. Although various numerical methods exist for the numerical inversion of the Laplace transform, the Fixed Talbot method shows the best performance based on previous tests. In the Talbot algorithm, it is necessary to adjust the parameters M and N to ensure convergence of the method; previous studies suggest a sufficient condition for convergence when the M/N ratio is within the interval $(5, 19)$. To investigate the suggested convergence interval, the numerical inversion of the Laplace transform via Fixed Talbot was applied to an initial-boundary value problem for the wave equation, and a comparison was made between the numerical and exact solutions through the maximum absolute error. The results suggest the conservation of the proposed convergence interval for this type of partial differential equation.

Keywords

Inverse Laplace Transform • Numerical Convergence • Parameter Values • Fixed Talbot Algorithm

Resumo

Este trabalho examina a inversão numérica da Transformada de Laplace via Talbot Fixo em um problema de valores iniciais de contorno, para a equação de onda. Mesmo existindo vários métodos numéricos de inversão numérica da transformada de Laplace, o Talbot Fixo é o que tem o melhor desempenho, com base em testes já realizados. No algoritmo de Talbot é necessário ajustar os parâmetros M e N para garantir a convergência do método, estudos anteriores sugerem condição suficiente de convergência quando a razão M/N fica no intervalo $(5, 19)$. Com o objetivo de investigar o intervalo de convergência sugerido, foi aplicado a inversão numérica da transformada de Laplace via Talbot Fixo em um problema de valores iniciais e de contorno, para equação de onda e realizada a comparação

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entre a solução numérica e exata através do máximo erro absoluto. Os resultados sugerem a conservação do intervalo de convergência proposto para esse tipo de equação diferencial parcial.

Palavras-chave

Transformada de Laplace Inversa • Convergência Numérica • Valores dos Parâmetros • Algoritmo de Talbot Fixo

1 Introduction

When seeking the solution of an equation via the Laplace transform [1,2], it is necessary to apply the inverse Laplace transform, which brings the result from the Laplace space back to the initial working space. Although this method greatly facilitates the solution of complex equations, performing the inversion of the transform analytically can often be difficult or even impossible. A solution to this issue is to use a numerical inversion method.

There are many methods for the numerical inversion of the Laplace transform, such as Dubner-Abate, Graver-Stehfest, Euler, and Fixed Talbot, as noted in Ref. [3], where it is also observed that the Fixed Talbot method [4] stands out in terms of accuracy and computational time.

In the application of the Fixed Talbot algorithm, it is necessary to adjust its parameters M (the number of terms in the sum formula of the algorithm) and N (the component of the fixation parameter r) to achieve convergence of the numerical method. This convergence does not always occur immediately, and multiple tests are sometimes required. One way to address this problem is to use the M/N ratio within the interval (5,19), as suggested in Ref. [5], after testing various benchmark functions and reinforced in Ref. [6] in solving a problem for the heat equation. In this work, we conduct a similar study for the wave equation, which models, for example, the mechanical vibrations of strings and bars [7]. It will be applied to a wave equation model to verify the validity of the (5, 19) interval for M/N, as has been done in other equations, as noted in Ref. [6].

2 Materials and Methods

As seen in Ref. [4], one of the pioneers in using numerical methods for inverting the Laplace transform $F(s) = \mathscr{L}[f(t)]$ was Talbot, by deforming the contour B of the Bromwich integral:

$$f(t) = \frac{1}{2\pi i}\int_B e^{ts}F(s)\,ds\,. \tag{1}$$

where B is a vertical line defined by $s = r + iy$, $-\infty < y < \infty$, and r has a fixed value chosen so that all singularities of the transform lie to its left.

The formula for the inversion of the Laplace transform by Talbot is as follows:

$$f(t) \approx \frac{r}{M}\left\{\frac{1}{2}F(r)\,e^{rt} + \sum_{k=1}^{M-1} Re\left[e^{ts(\theta_k)}F\left(s\left(\theta_k\right)\right)\left(1 + i\sigma\left(\theta_k\right)\right)\right]\right\}, \tag{2}$$

where $\theta_k = k\pi/M$, $\sigma(\theta) = \theta + \left(\theta\cot(\theta) - 1\right)\cot(\theta)$ e $r = 2M/(Nt)$, M is the number of terms to be summed and r is the fixation parameter. Details on how to go from Eq. (1) to Eq. (2) can be found in Ref. [4] for $N = 5$.

After testing values of M and N in Refs. [5,6] to achieve better convergence for the algorithm, it was conjectured that there is convergence for $5 < M/N < 19$.

The convergence characteristic was evaluated by applying the maximum absolute error between the result $f_{TF}(t)$ provided by the Fixed Talbot algorithm and the exact value $f(t)$ of the corresponding function:

$$Error = \max_{t\in[a,\,b]} |f(t) - f_{TF}(t)|\,, \tag{3}$$

assuming conservatively that the accuracy is acceptable when error (3) is less than 1.

3 A Problem for the Wave Equation

With the objective of solving an IBVP (initial-boundary value problem) for the homogeneous wave equation over an infinite homogeneous medium with unit characteristic speed, initial spatial profile $u_0(x)$, and initial velocity

$v_0(x)$, with $u_0, v_0 \in C^2([a,b])$, and fixed endpoints. The unknown $u \in C^2([a,b] \times R_+)$ is the spatial profile of the wave in $[a,b] \ni x$ at the time instant $t \in R_+$.

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (a,b) \times \mathbb{R}_+^* \\ u(x,0) = u_0(x), & x \in [a,b] \\ u_t(x,0) = v_0(x), & x \in [a,b] \\ u(a,t) = u(b,t) = 0, & t \in \mathbb{R}_+ \end{cases}. \tag{4}$$

The solution of the IBVP in Eq. (4) using the Laplace transform is presented below. Let $U(x,s) = \mathscr{L}[u(x,t)]$. Then, applying $\mathscr{L}[\cdot]$ to the wave equation of the IBVP yields

$$\mathscr{L}[u_{tt}] - \mathscr{L}[u_{xx}] = \mathscr{L}[0], \tag{5}$$

so we have

$$s^2 U(x,s) - s u(x,0) - u_t(x,0) - U_{xx}(x,s) = 0, \tag{6}$$

from which, applying the initial conditions of the IBVP, results in the ODE

$$s^2 U(x,s) - s u_0(x) - v_0(x) - U_{xx}(x,s) = 0, \tag{7}$$

which can be written in canonical form as

$$U_{xx}(x,s) - s^2 U(x,s) = -s u_0(x) - v_0(x). \tag{8}$$

Eq. (8) is complemented by the boundary conditions $U(a,s) = U(b,s) = 0$, obtained by applying $\mathscr{L}[\cdot]$ to the boundary conditions of the IBVP in Eq. (4). Thus, we have the following BVP in the Laplace space for the second-order ODE in x, linear and non-homogeneous in Eq. (8), subject to homogeneous Dirichlet boundary conditions:

$$\begin{cases} U_{xx} - s^2 U(x,s) = F(x,s) \\ U(a,s) = U(b,s) = 0 \end{cases}, \tag{9}$$

where the non-homogeneity of the ODE is $F(x,s) = -\left(s u_0(x) + v_0(x)\right)$.

The solution of the BVP in Eq. (9) obtained by the method of variation of parameters [8] is

$$U(x,s) = K_1(x,s)\, e^{-sx} + K_2(x,s)\, e^{sx}, \tag{10}$$

where

$$K_1(x,s) = -\frac{1}{2s}\int e^{sx} F(x,s)\, dx + C_1(s) \ , \tag{11}$$

and

$$K_2(x,s) = \frac{1}{2s}\int e^{-sx} F(x,s)\, dx + C_2(s) \ , \tag{12}$$

with $C_1(s)$ and $C_2(s)$ obtained by applying the boundary conditions of the BVP. Thus, the solution of the original IBVP in Eq. (4) follows from applying the inverse Laplace transform $u(x,t) = \mathscr{L}^{-1}[U(x,s)]$ to Eq. (10), taking into account Eqs. (11) and (12). Note that, given the general complexity of the solution $U(x,s)$, it may be necessary to apply a numerical inversion algorithm for the Laplace transform to obtain the solution $u(x,t) = \mathscr{L}^{-1}[U(x,s)]$ of the original IBVP. As stated above, in this work, the Fixed Talbot algorithm presented in the previous section is employed. The following section presents the computational results of applying the Fixed Talbot algorithm to a specific problem, comparing it with the exact analytical solution through the maximum absolute error to investigate its convergence regarding the variation of the M/N ratio of its parameters M and N.

4 Results and Discussions

Now consider the particular case of the IBVP in Eq. (3) on the interval $[a, b] = \left[\frac{1}{2}, \frac{2}{3}\right]$, with initial conditions $u_0(x) = cos(\pi x)$ and $v_0(x) = 0$. Thus, $F(x, s) = -\cos(\pi x)$. Using the method shown previously, we obtain the following solution in the Laplace space:

$$U(x, s) = \frac{cos(\pi x)}{s^2+\pi^2}, \tag{13}$$

and by applying the inverse Laplace transform, we have the exact solution of the IBVP as

$$u(x, t) = \mathscr{L}^{-1}[U(x,s)] = \frac{1}{\pi}\cos(\pi x)\,\text{sen}\,(\pi t). \tag{14}$$

Based on the exact solution in Eq. (14), whose graph is presented in Fig. 1, we will apply the numerical inversion of Eq. (13) using the Fixed Talbot algorithm from Eq. (2) to obtain the computational assessment of the problem's solution and make a comparison between the two solutions (exact and numerical) as the parameters M and N, and consequently the M/N ratio, vary. This aims to identify the most suitable interval for convergence based on the maximum absolute error between the exact solution and the numerical solution via Fixed Talbot, implemented in Python [9].

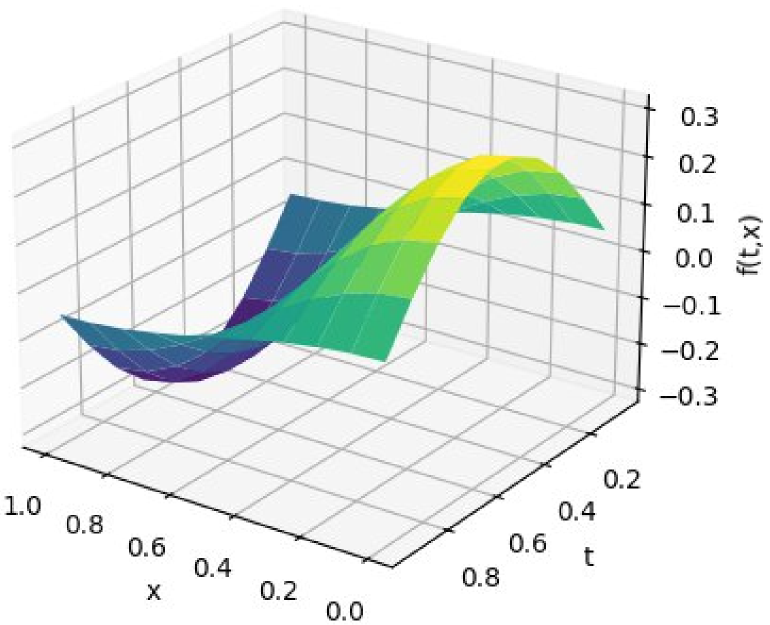

Figure 1: Surface of the exact solution $u(x, t)$ in Eq. (14).

The evolution of the maximum absolute error between the exact and numerical solutions as described above, varying the M/N ratio from 1 to 20 while keeping $N = 7$, is presented in Fig. 2. Note that errors greater than 1 start to appear when the ratio is 19, where the maximum absolute error is nearly 1.97, and as the ratio continues to increase, the error grows exponentially.

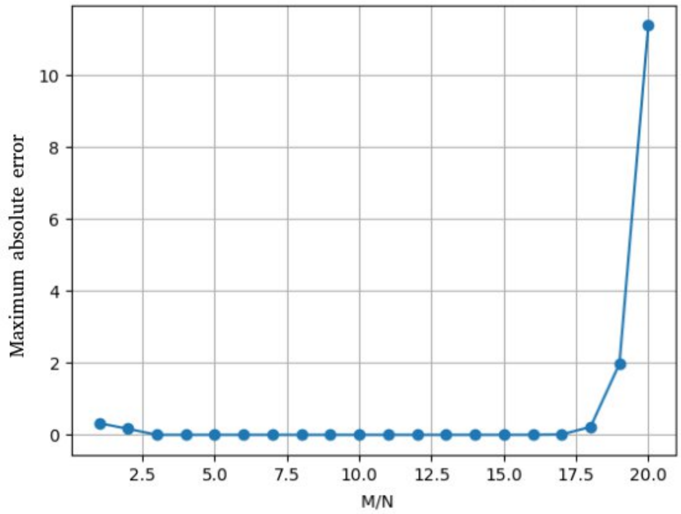

Figure 2: Error as a function of the ratio M/N.

5 Conclusions

This work briefly presented recent efforts aimed at achieving better convergence of the numerical inversion method of the Laplace transform via Fixed Talbot. In this case, it was applied to an initial-boundary value problem for the wave equation, where convergence was observed within the suggested interval. The application and evaluation of the convergence interval in other partial differential equations are left for future work.

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