# An Analytical Diffusion Model and Its Application to Soil Heat Conduction☆

## Modelo Analítico da Difusão e Sua Aplicação à Condução do Calor no Solo

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**Abstract**

This paper presents an alternative approach to solving an initial and boundary value problem (IBVP) related to the one-dimensional non-homogeneous diffusion equation with constant coefficients, applied to heat conduction in soil. Unlike the traditional Fourier separation of variables, the proposed approach is based on the fundamental solution (Green's function), the antisymmetric extension of the initial condition, and the principles of Duhamel and superposition. The methodology involves analyzing the Cauchy problem for the heat equation in unbounded and semi-infinite domains using variable transformation and integration techniques. Preliminary results indicate that this technique can be effectively applied in future studies on ground-air heat exchangers, aimed at thermal comfort in indoor environments. This study is an important initial step towards developing more efficient and accurate solutions that describe soil behavior.

**Keywords**

Heat equation • Duhamel's principle • Green's function

**Resumo**

Este trabalho apresenta uma abordagem alternativa para a solução de um problema de valores iniciais e de contorno (PVIC) relacionado à equação de difusão unidimensional não homogênea com coeficientes constantes, aplicada à condução de calor no solo. Ao contrário do método tradicional de separação de variáveis de Fourier, a abordagem proposta baseia-se na solução fundamental (função de Green), na extensão antissimétrica da condição inicial, e nos princípios de Duhamel e de superposição. A metodologia envolve a análise do problema de Cauchy para a equação do calor em domínios ilimitados e semi-infinitos, utilizando técnicas de transformação de variáveis e integração. Os resultados preliminares indicam que essa técnica pode ser aplicada efetivamente em estudos futuros sobre trocadores de calor solo-ar, visando conforto térmico em ambientes fechados. Este estudo é um passo inicial importante para o desenvolvimento de soluções mais eficientes e precisas que descrevem o comportamento do solo.

**Palavras-chave**

Equação do calor • Princípio de Duhamel • Função de Green

☆This article is an extended version of the work presented at the Joint XXVII ENMC National Meeting on Computational Modeling, XV ECTM Meeting on Science and Technology of Materials, held in Ilhéus–Brazil,from October 1st to 4th,2024.

# 1 Introduction

The analysis of diffusive phenomena is essential for understanding various natural and industrial processes. This study proposes an alternative approach to obtain a closed-form expression for the exact solution of an initial and boundary value problem for the one-dimensional non-homogeneous diffusion equation with constant coefficients in a semi-infinite medium. Unlike the Fourier separation of variables method used [1], the adopted approach is based on the fundamental solution, the antisymmetric extension of the initial condition, and the principles of Duhamel and superposition, as described in Ref. [2].

Partial differential equations often arise from conservation laws, which state that certain physical properties of an isolated system remain invariant over time. Diffusion, as a transport phenomenon, describes how a conserved quantity, such as heat or mass, distributes itself in space over time due to concentration differences [2].

The amount of a conserved quantity is described by $u(x,t)$, where $u$ represents the density at position $x$ and time $t$. The movement of this quantity is represented by the flux, which is crucial to determine the system's dynamics. The flux density at a specific point in space provides a detailed view of the transfer of the quantity across an infinitesimally small surface.

The relevance of this study lies in the practical application of the obtained solutions to real-world problems, such as in spatial ecology and evolutionary biology [3], building physics [4], heat conduction [5], atmospheric phenomena [6], meteorology [7], pollutant dispersion [8, 9], and image restoration and interpolation [10]. The presented methodology broadens the theoretical understanding of diffusive phenomena and provides tools for applied engineering, enhancing the efficiency of thermal systems in various contexts.

Moreover, as described by Feynman in his famous *Lectures on Physics* [11], diffusive phenomena play a crucial role in various scientific fields. Diffusion is fundamental in the analysis of processes such as heat transfer in solid materials, electrical conduction in semiconductors, particle propagation in heterogeneous media, and dielectric polarization. Feynman also emphasizes the importance of diffusion in chemical processes, such as the mixing of substances in solutions. These examples illustrate the versatility of diffusive phenomena and the relevance of understanding them to advance multiple areas of scientific and technological knowledge.

This work investigates the solution of the heat diffusion equation in soil with a zero-source term and non-homogeneous conditions to compare the efficiency of resolution methods. The methods addressed, especially the approach based on the fundamental solution, proved to be effective and precise. The computational implementation of the solutions, using Python [12] and scientific libraries such as NumPy, SciPy, and Matplotlib, enabled a detailed analysis and visualization of diffusive phenomena. This comparative analysis highlights the effectiveness of the proposed approach over other resolution methods for diffusive phenomena.

In Section 2, we discuss the analysis of partial differential equations in different domains. Both theoretical and computational approaches are used to solve classical and non-classical diffusion problems. Section 3 presents the obtained results, focusing on heat diffusion in soil and the computational modeling of the studied phenomena, simulating temperature distribution under different conditions. Finally, Section 4 discusses the results, highlighting the theoretical and practical implications of the presented solutions.

# 2 Methodology

## 2.1 Cauchy Problem for the Diffusion Equation in Unbounded Domains

The choice of spatial domain is crucial in solving problems, as it can be bounded or unbounded. The preference for a specific domain type depends on the problem's characteristics. According to Ref. [2], although problems in bounded domains are common, unbounded domains offer advantages, such as simplifying the problem formulation and reducing the complexity of boundary conditions. Understanding the categories of partial differential equations and considering the spatial domain are fundamental for effective problem-solving. This initial approach can simplify analysis and provide a solid foundation for further investigations.

Consider the following IVP for the homogeneous diffusion equation with non-homogeneous initial conditions:

$$u_t - ku_{xx} = 0,\ (x,t) \in \mathbb{R} \times \mathbb{R}_+^*, \qquad u(x,0) = u_0(x),\ x \in \mathbb{R}, \tag{1}$$

where $u$ is the density or concentration of the diffused quantity, and $k$ is the diffusivity of the medium.

Here, $k > 0$ is the proportionality constant. The negative sign ensures that if $u_x < 0$, then $\phi$ will be positive, meaning the flux moves to the right; if $u_x > 0$, then $\phi$ will be negative, meaning the flux moves to the left. The flux occurs in the direction of the gradient. The constant $k$ is called the diffusion coefficient, measured in length squared per unit of time. When the constitutive equation is substituted into the conservation law, we obtain a simple model equation.

In the context of thermal conduction, the problem represented by Eq. (1) models the propagation of heat in an infinite rod with an initial temperature distribution $u_0(x)$. In general, it models the density variation in diffusive processes. When $x \in \mathbb{R}$, there are no boundary conditions, although sometimes additional conditions are required.

To solve this type of Cauchy problem, it is necessary to consider an auxiliary problem where the initial condition is a finite jump, also known as the Heaviside function, defined as $H(x)$, the unit step function:

$$w_t - kw_{xx} = 0,\ (x,t) \in \mathbb{R} \times \mathbb{R}_+^*, \qquad w(x,0) = w_0 H(x),\ x \in \mathbb{R}. \tag{2}$$

The variable $w$ and the constant $w_0$ represent temperatures in Kelvin ($K$). The variable $x$ denotes position in meters ($m$), while $t$ represents time in seconds ($s$). In this context, $k$ is the thermal diffusivity of the medium, measured in square meters per second ($m^2/s$). The dimensionless quantities of the problem are given by $w/w_0$ and $x/\sqrt{4kt}$.

The expected solution is of the form $w/w_0 = f(x/\sqrt{4kt})$. Substituting this into problem described by Eq. (2), we obtain a first-order linear homogeneous ODE for $f'$ with respect to its argument, which, taking the initial condition into account, yields the solution:

$$w(x,t) = \frac{w_0}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right), \tag{3}$$

where the Gaussian error function is defined as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2}\, dr. \tag{4}$$

Moreover, if $w$ is a solution of the heat equation, then $w_x$ is also a solution. Therefore, for $w_0 = 1$, we have:

$$w(x,t) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \Rightarrow w_x = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}} = G(x,t). \tag{5}$$

The function $G(x,t)$ obtained can be called the heat kernel, the Green's function for the heat equation, or the fundamental solution of the heat equation. This function represents the temperature distribution resulting from a point source of heat located at $x = 0$, which initially transfers one unit of heat, $G(x,0) = w_0 = 1$. The function $G(x-y,t)$ represents the temperature distribution when the heat source is located at $x = y$.

## 2.2 PDEs in Semi-Infinite Domains

Consider the following IBVP for the non-homogeneous diffusion equation:

$$u_t - ku_{xx} = 0,\ (x,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \quad u(0,t) = 0,\ t \in \mathbb{R}_+, \quad u(x,0) = u_0(x),\ x \in \mathbb{R}_+, \tag{6}$$

where $u_0(0) = 0$ ensures compatibility with the boundary condition. According to Ref. [2], the solution to problem indicated in Eq. (6) for $x > 0$ and $t > 0$ will be derived from the solution (5) of the corresponding Cauchy problem (1) for $x \in \mathbb{R}$. Specifically, an auxiliary variable $v(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ will be introduced, where $v = u$ for $x > 0$ in the solution $u$ of problem (6).

Let $v_0$ be the odd extension of the initial condition $u_0$ to the entire real line $\mathbb{R}$, defined as $v_0(x) = -u_0(-x)$ for $x < 0$. This allows us to formulate the Cauchy problem with $v_t - kv_{xx} = 0$ for $x \in \mathbb{R}$ and $t > 0$, where the solution $v(x,t)$ satisfies the initial condition $v(x,0) = v_0(x)$ and is given by:

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{+\infty} v_0(y) G(x-y,t)\, dy \\ &= \int_{-\infty}^{0} v_0(y) G(x-y,t)\, dy + \int_0^{+\infty} v_0(y) G(x-y,t)\, dy \\ &= \int_0^{+\infty} u_0(y) \left[G(x-y,t) - G(x+y,t)\right] dy, \quad x \in \mathbb{R}. \end{aligned} \tag{7}$$

Thus, the solution $u$ of the original initial and boundary value problem (6), obtained by restricting the solution $v$ in (7) to $\mathbb{R}_+$, is:

$$u(x,t) = \int_0^{+\infty} u_0(y) \left[G(x-y,t) - G(x+y,t)\right] dy, \quad x > 0,\ t > 0. \tag{8}$$

## 2.3 Duhamel's Principle for Problems with Non-Zero Source Term

Consider the IVP for the non-homogeneous PDE with homogeneous initial condition:

$$u_t - ku_{xx} = f(x,t),\ x \in \mathbb{R},\ t > 0, \quad u(x,0) = 0,\ x \in \mathbb{R}. \tag{9}$$

From Duhamel's principle, the solution is:

$$u(x,t) = \int_0^t u^{(\tau)}(y,\tau)(x, t-\tau)d\tau, \tag{10}$$

where $u^{(\tau)}(x,t)$ is the solution of the Cauchy problem for the homogeneous diffusion equation with a non-homogeneous initial condition given by the source term of the original problem:

$$u_t^{(\tau)} - ku_{xx}^{(\tau)} = 0,\ x \in \mathbb{R}, \qquad u^{(\tau)}(x,0) = f(x,\tau), \tag{11}$$

whose solution is

$$u^{(\tau)}(x,t) = \int_{-\infty}^{+\infty} f(y,\tau)G(x-y,t)dy, \tag{12}$$

from which the solution of the original problem is

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} f(y,\tau)G(x-y,t-\tau)dyd\tau. \tag{13}$$

## 2.4 Solution of the Non-Homogeneous Diffusion Equation with Non-Homogeneous Initial Condition

Consider the IVP for the non-homogeneous diffusion equation with a non-homogeneous initial condition:

$$u_t - ku_{xx} = f(x,t),\ x \in \mathbb{R}, t > 0, \qquad u(x,0) = u_0(x),\ x \in \mathbb{R}. \tag{14}$$

Given the linearity of the diffusion equation, the solution to problem (14) can be obtained by the principle of superposition, setting $u = u^{(1)} + u^{(2)}$ based on the solutions of the auxiliary IVPs

$$u_t^{(1)} - ku_{xx}^{(1)} = f(x,t),\ x \in \mathbb{R}, t > 0, \qquad u^{(1)}(x,0) = 0,\ x \in \mathbb{R}, \tag{15}$$

and

$$u_t^{(2)} - ku_{xx}^{(2)} = 0,\ x \in \mathbb{R}, t > 0, \qquad u^{(2)}(x,0) = u_0,\ x \in \mathbb{R}. \tag{16}$$

Thus, the solution to the original problem is:

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} f(y,\tau)G(x-y,t-\tau)dyd\tau + \int_{-\infty}^{+\infty} u_0(y)G(x-y,t)dy. \tag{17}$$

Consider the initial and boundary value problem for the diffusion equation with a zero source term and a non-homogeneous boundary condition:

$$u_t - ku_{xx} = 0,\ x > 0, t > 0, \quad u(0,t) = u^0(t),\ t > 0, \quad u(x,0) = u_0(x),\ x > 0. \tag{18}$$

To homogenize the boundary condition, it is convenient to introduce a new variable $v(x,t) = u(x,t) - u^0(t)$, such that:

$$v_t - kv_{xx} = -u_t^0,\ x > 0, t > 0, \quad v(0,t) = 0,\ t > 0, \quad v(x,0) = v_0(x),\ x > 0, \tag{19}$$

where $v_0(x) = u_0(x) - u^0(0)$. Given the linearity of the diffusion equation, the solution will be the superposition of the solutions to the auxiliary problems, hence:

$$v_t^{(1)} - kv_{xx}^{(1)} = -u_t^0,\ x > 0, t > 0, \quad v^{(1)}(0,t) = 0,\ t > 0, \quad v^{(1)}(x,0) = 0,\ x > 0, \tag{20}$$

and

$$v_t^{(2)} - kv_{xx}^{(2)} = 0,\ x > 0, t > 0, \quad v^{(2)}(0,t) = 0,\ t > 0, \quad v^{(2)}(x,0) = v_0(x),\ x > 0. \tag{21}$$

The solution for the first auxiliary problem will be given by Duhamel's principle, so:

$$v^{(1)}(x,t) = -\int_0^{+\infty} \int_0^t u_t^0(t)[G(x-y,t-\tau) - G(x+y,t-\tau)]d\tau dy. \tag{22}$$

The solution for the second auxiliary Cauchy problem will be:

$$v^{(2)}(x,t) = \int_0^{+\infty} v_0(x)[G(x-y,t) - G(x+y,t)]dy. \tag{23}$$

Thus, the solution to the original problem is:

$$\begin{aligned} u(x,t) = &- \int_0^{+\infty}\int_0^t u_t^0(t)[G(x-y,t-\tau) - G(x+y,t-\tau)]d\tau dy \\ &+ \int_0^{+\infty} u_0(y)[G(x-y,t) - G(x+y,t)]dy + u^0(t) \end{aligned}. \tag{24}$$

# 3 Results

## 3.1 Heat Diffusion in Soil

Heat conduction in soil is governed by the IBVP (18). This equation varies only with respect to time and depth $x = z$, given certain assumptions. First, the soil is under natural conditions. Second, soil stratification increases with depth. Third, there is no internal heat generation (i.e., $f(z,t) = 0$). According to [1], at the surface, where $z = 0$, the boundary condition for the heat diffusion equation is

$$u(0,t) = u^0(t) = T_0 + \theta_0 \sin(\omega t),\ t > 0, \tag{25}$$

where $T_0$ is the average temperature value, $\theta_0$ is the amplitude of variation, and $\omega$ is the frequency. On the other hand, the initial condition

$$u(z,0) = u_0(z) = T_0 - \theta_0 e^{-\sqrt{\frac{\omega}{2k}}z} \sin\left(\sqrt{\frac{\omega}{2k}}z\right),\ z > 0, \tag{26}$$

follows by setting $t = 0$ in the solution obtained in [1] using the method of separation of variables:

$$\overline{u}(z,t) = T_0 + \theta_0 e^{-\sqrt{\frac{\omega}{2k}}z} \sin\left(\omega t - \sqrt{\frac{\omega}{2k}}z\right). \tag{27}$$

Thus, with these considerations, the IBVP in question is

$$u_t - k u_{zz} = 0,\ z > 0, t > 0, \quad u(0,t) = u^0(t),\ t > 0, \quad u(z,0) = u_0(z),\ z > 0, \tag{28}$$

where $u^0(t)$ and $u_0(z)$ are given by (25) and (26), respectively.

Defining $v(z,t) = u(z,t) - u^0(t)$ to homogenize the boundary condition, the following IBVP with a non-homogeneous equation is obtained:

$$v_t - k v_{zz} = -\omega\theta_0 \cos(\omega t),\ z > 0, t > 0, \quad v(0,t) = 0,\ t > 0, \quad v(z,0) = v_0(z),\ z > 0, \tag{29}$$

where $v_0(z) = u_0(z) - u^0(t)$. Given the linearity of the heat equation and the superposition principle, the solution is $v = v^{(1)} + v^{(2)}$, where $v^{(1)}$ and $v^{(2)}$ are solutions of the following auxiliary IBVPs:

$$v_t^{(1)} - k v_{zz}^{(1)} = -\omega\theta_0 \cos(\omega t),\ z > 0, t > 0, \quad v^{(1)}(0,t) = 0,\ t > 0, \quad v^{(1)}(z,0) = 0,\ z > 0. \tag{30}$$

and

$$v_t^{(2)} - k v_{zz}^{(2)} = 0,\ z > 0, t > 0, \quad v^{(2)}(0,t) = 0,\ t > 0, \quad v^{(2)}(z,0) = v^0(z),\ z > 0. \tag{31}$$

Since $u(z,t) = v(z,t) + u^0(t)$, the solution to the IBVP (28) is

$$\begin{aligned} u(z,t) = &-\omega\theta_0 \int_0^{+\infty}\int_0^t \cos(\omega\tau)\,[G(z-y,t-\tau) - G(z+y,t-\tau)]\,d\tau dy \\ &- \int_0^{+\infty} \theta_0 e^{-\sqrt{\frac{\omega}{2k}}y} \sin\left(\sqrt{\frac{\omega}{2k}}y\right)[G(z-y,t) - G(z+y,t)]\,dy \\ &+ T_0 + \theta_0 \sin(\omega t) \end{aligned} \tag{32}$$

### 3.2 Computational Modeling

Through the solution (32), the analytical solution was implemented to provide a detailed representation of heat propagation over time and depth, comparing it to alternative approaches, such as the method used in [1].

To implement the analytical solution, the Python programming language was used, taking advantage of the libraries NumPy, SciPy, and Matplotlib for numerical calculations, integration, and visualization, respectively. With the developed algorithms, it was possible to perform three types of analyses.

In Fig. 1, a comparison is presented between equations (27) and (32), aimed at analyzing temperature variation over time. The parameters $\theta_0 = 6.28$, frequency $\omega = 2\pi/365$, and diffusivity constant $k = 0.057\,\mathrm{mm}^2/s$ were considered. It was observed that the temperature variation coincides at all points, with the same behavior observed for temperature variation with respect to depth. The solution $u(z,t)$ is represented by circular markers, while the solution in [1] is represented by continuous curves.

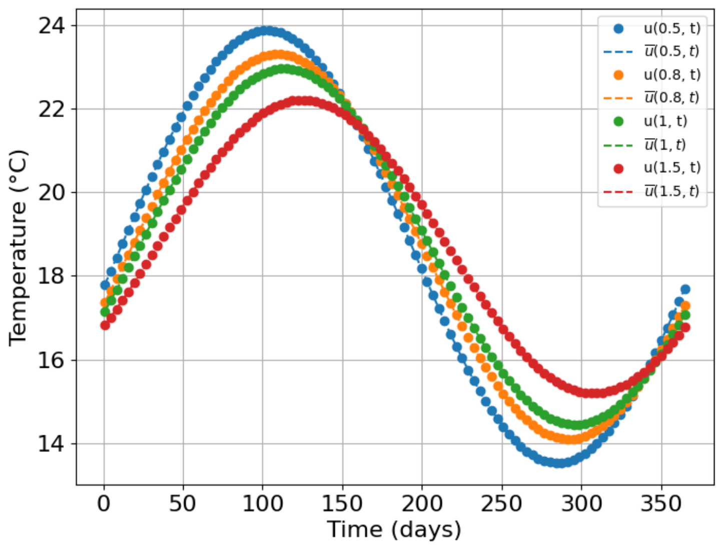


Figure 1: Comparison between the solutions $u(t)$ and $\overline{u}(t)$.

In Fig. 2, it can be seen that as depth increases, the temperature tends towards the average value of 18 ºC.

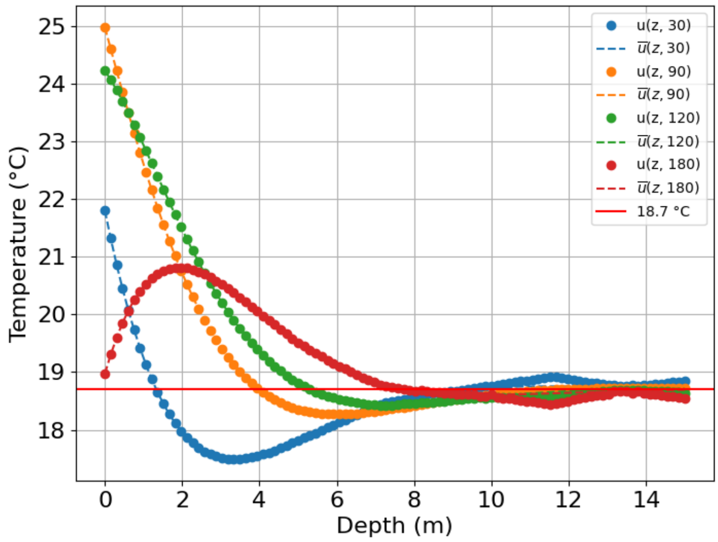


Figure 2: Comparison between the solutions $u(z)$ and $\overline{u}(z)$.

The solution can also be obtained in $\mathbb{R}^3_+$, where it is possible to observe the simultaneous variation of temperature with respect to time and depth, as shown in Fig. 3.

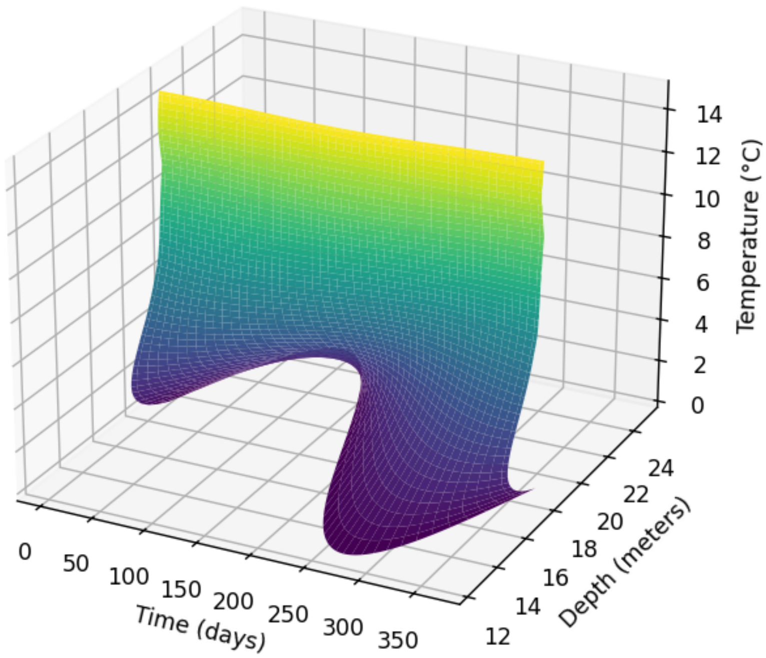


Figure 3: Temperature variation $u(z, t)$.

The visualization of the absolute error variation, given by $e(z, t) = |u(z, t) - \overline{u}(z, t)|$, indicates that $0 \leq e(z, t) \leq 0.1975$, as shown in Fig. 4.

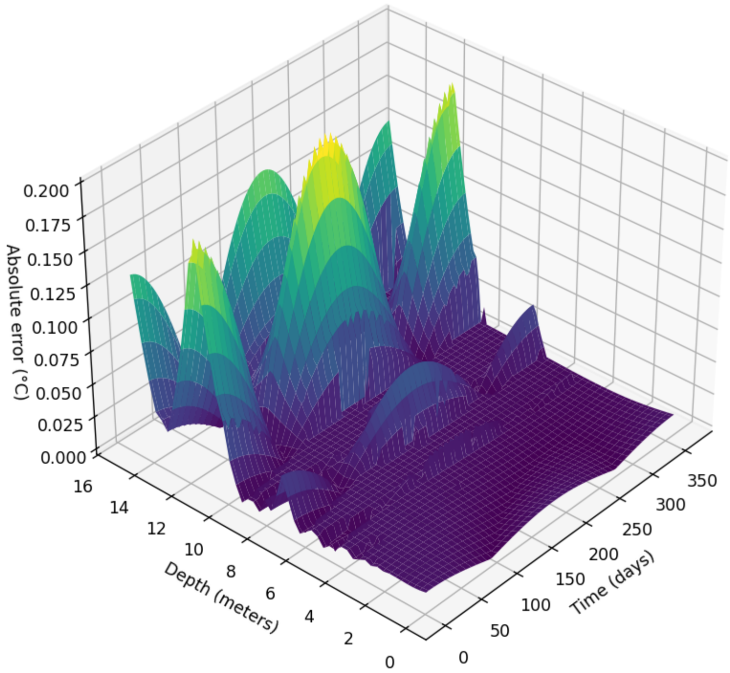


Figure 4: Error variation $e(z, t)$.

## 4 Conclusions

This study demonstrates the effectiveness of the approach using Duhamel's principle in modeling heat diffusion in soil. The obtained analytical solution provides a detailed representation of heat propagation, allowing for a deeper understanding of the studied phenomenon. Additionally, the comparison with the Fourier method, as used in [1], highlights the advantages and disadvantages of each approach in solving heat transfer problems.

Although both methods were effective in modeling heat diffusion in soil, the choice between them may depend on the specific characteristics of the problem and the available computational resources. The Duhamel method may be more suitable in situations where boundary conditions and medium properties vary over time, while the Fourier method may be preferable in problems with symmetry and well-defined boundary conditions.

In summary, this study contributes to advancing knowledge in the area of heat transfer in soil, providing a solid foundation for future research and practical applications in various fields of engineering and environmental sciences.

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