VETOR

Parameter N Influence on Fixed-Talbot Algorithm for Laplace Transform Numerical Inversion☆

Influência do Parâmetro N no Algoritmo de Talbot-Fixo para a Inversão Numérica da Transformada de Laplace

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Abstract

In this paper, Fixed-Talbot method computational aspects are explored for Laplace Transform numerical inversion and its efficiency in the treatment of a set of elementary functions of an exponential, oscillatory and logarithmic nature, based on the influence investigation of free parameter N. The numerical results are compared to the analytical solution while calculating the absolute error. The best value for N was determined in each studied function class, where the method presents satisfactory results. It was observed that increasing the number of terms in the summation for approximation (beyond the optimal value) doesn't imply obtaining more refined results. In general, based on the data obtained, it was concluded that Fixed-Talbot method is efficient for the inversion of all classes of elementary functions evaluated in this article.

Keywords

Transformada de Laplace • Transformada Inversa • Métodos Numéricos • Talbot-Fixo

Resumo

Neste artigo, são explorados os aspectos computacionais do método de Talbot-Fixo para a inversão numérica da Transformada de Laplace e sua eficiência no tratamento de um conjunto de funções elementares de natureza exponencial, oscilatória e logarítmica, a partir da investigação da influência do parâmetro livre N. Os resultados numéricos são comparados à solução analítica, calculando-se o erro absoluto. O melhor valor para N, em cada classe de função estudada, nos quais o método apresenta resultados satisfatórios, foram determinados. Observou-se que aumentar o número de termos do somatório para a aproximação (além do valor ótimo) não implica em obter resultados mais refinados. De um modo geral, fundamentado nos dados obtidos, conclui-se que o método de Talbot-Fixo é eficiente para a inversão de todas as classes de funções elementares avaliadas neste trabalho.

Palavras-chave

Laplace Transform • Inverse Transform • Numerical Methods • Fixed-Talbot

1 Introduction

Laplace Transform is a powerful tool, generally applied to initial value problems resolution, involving physical phenomena such as electromagnetism [1], heat transfer [2]; as well as in areas such as hydrology [3], bio-medicine [4],

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among others.

However, depending on the solution format, in many applications it's impossible to invert Laplace Transform analytically, making the numerical approach the only viable one [5]. According to Abate and Valkó [6], it's estimated that there are more than 100 methods available in the literature for Laplace Transform numerical inversion, which can be classified into four categories known as (i) Fourier Series Expansion;(ii) Laguerre Function Expansion; (iii) Combination of Gaver Functionals; (iv) Bromwich Contour Deformation.

The non-existence of a universal method for inverting this technique makes the vast research in this area justifiable as it provides subsidies for a correct method to be used for each case, where the solution is known. Among the specifications, the processing time, numerical precision, and implementation difficulty have been evaluated, confirming that no method is superior to others in all aspects [7].

Within this context, in this study, we analyze the parameter N influence on the Fixed-Talbot (FT) method by performing tests on elementary transformed functions, whose analytical inverse is known.

The free parameter N indicates the number of terms used in the series that approximate the function to be transformed by the FT method. This parameter can vary in order to provide results with the smallest possible error.

We also evaluate the method as to its performance for Laplace Transform inversion of different classes of functions, as well as the implemented algorithm validation.

2 Fixed-Talbot Method

Abate and Valkó [6] denoted a variation, proposed by Talbot [8], which performs the inversion from the Bromwich integral by using Fixed-Talbot method.

The Fixed-Talbot algorithm can be simply implemented and it will always find a solution if it exists [9]. This technique has demonstrated stability in solving problems of electrochemical systems, in elementary functions tests ([6], [10], [11]), Müntz polynomials evaluation [1], viscoelastic waves [12] and heat conduction [13]. Thus, the contour deformation of Bromwich used here is

$$s(\theta) = r\theta(\cot(\theta) + i), -\pi < \theta < \pi. \tag{1}$$

After differentiating Eq. (1), we obtain

$$s'(\theta) = ir(1 + i\sigma(\theta)), \tag{2}$$

where

$$\sigma(\theta) = \theta + (\theta\cot(\theta) - 1)\cot(\theta). \tag{3}$$

After some simplifications, constants manipulation and integration limits adequacy, it comes to

$$f_{FT}(t) = \frac{r}{\pi}\int_0^{\pi} Re\left[e^{ts(\theta)}F(s(\theta))(1 + i\sigma(\theta))\right] d\theta, \tag{4}$$

where t is a dimensionless variable, $Re[\ ]$ matches the real part of the argument and s is the parameter in Laplace transform given by Eq. (1).

Applying the Trapezoidal Rule, using N points equally spaced, with step size $\frac{\pi}{N}$ [14], $N \in \mathbb{N}$, and $\theta_k = \frac{k\pi}{N}$, where $k = 1, 2, ..., N-1$ [9], the Eq. (4) can be approximated by

$$f_{FT}(t) \approx \frac{r}{N}\left\{\frac{1}{2}F(r)e^{rt} + \sum_{k=1}^{N-1} Re\left[e^{ts(\theta_k)}F(s(\theta_k))(1 + i\sigma(\theta_k))\right]\right\}. \tag{5}$$

It is worth remembering that the Trapezoidal Rule is a numerical integration method where the integration interval is divided into N sub-intervals (partition) and, in each sub-interval, the integrand is approximated by a linear function using Lagrange polynomials.

From the obtained results in numerical experiments [6], it is defined that

$$r = \frac{2N}{5t}. \tag{6}$$

In this way, Fixed-Talbot approximation of $f(t)$ becomes dependent on only free parameter N, which represents the number of terms of the summation in Eq. (5).

3 Results and Discussions

In this section, we discuss the numerical results obtained through Laplace Transform numerical inversion using Fixed-Talbot (FT) technique. To carry out the tests and analysis of the inversion method, five elementary functions were used (as can be seen in Table 1). The purpose of these tests is to explore and evaluate the performance of the method for inverting these five classes of functions and analyze the parameter N influence.

In particular, the influence on function $f_4(t)$, where $\gamma \approx 0.5772156649$ is the Euler-Mascheroni constant (also called Euler's constant) [15].

3.1 Testing functions

For the validation tests, Octave 5.2.0 free software was used on a computer with a Windows Home Single Language operating system, Intel(R) Core(TM) i3-5005U processor at 2.00 GHz and memory (RAM) at 4.00 GB. Table 1 presents the test functions and their respective Analytic Inverse Transforms.

Table 1: Laplace Transform and its respective inverse.

$F(s)$	$f(t)$
$F_1(s) = \dfrac{1}{s^2+1}$	$f_1(t) = \sin(t)$
$F_2(s) = \dfrac{1}{(s+1)(s+2)}$	$f_2(t) = e^{-2t}(e^t - 1)$
$F_3(s) = \dfrac{1}{s^2+s+1}$	$f_3(t) = \dfrac{2\sqrt{3}}{3} e^{-0.5t} \sin\left(\dfrac{\sqrt{3}}{2} t\right)$
$F_4(s) = -\dfrac{\ln s}{s}$	$f_4(t) = \ln t + \gamma$
$F_5(s) = \dfrac{1}{(s+1)^2}$	$f_5(t) = te^{-t}$

The Tables 2-6 present the absolute errors and the parameters values that were used here. Figures 1-5 show a comparison between the Analytical Inverse Transform and the Numerical Inverse Transform obtained by FT method for each value of N. The functions were examined for the values of $N \in \mathbb{N}$, where $N = 1, 2, 3, ..., 70$, indicated by the literature ([6], [10], [12]), and it was established as satisfactory results, where the absolute error order is lower than 10^{-8}. The best value of N, which represents the number of terms in the summation for the approximation of $f(t)$ in each case, was determined from the analysis of the mean absolute error ($\overline{E}_{abs}$), defined as

$$\overline{E}_{abs} = \frac{1}{N_t} \sum_{i=1}^{N_t} E_{abs}(f(t_i)) \tag{7}$$

where N_t is the number of t values used to generate each profile on Tables 2-6 and $E_{abs}(f(t_i))$ is the absolute error estimated between the numerical and analytical inversions for each functions f at each point t_i ([16]).

Due to the parameter r definition in terms of the variable t (Eq. (6)), it was agreed that all the tests started from $t = 10^{-4}$.

3.2 The inversion of $F_1(s)$

Figure 1 and Table 2 demonstrate the numerical inversion tests for Laplace Transform through the Fixed-Talbot method for the function $F_1(s) = \dfrac{1}{s^2+1}$. It can be noted that FT method shows the best results using $N = 30$, producing absolute error of predominant order between 10^{-12} and 10^{-13}.

On the other hand, small values of N ($N = 5$ for example) lead to a great loss on the results generation, including adjustment prevention between the analytical and numerical profiles. In the same way, increasing N value without limit causes an error accumulation that prevents to obtain satisfactory results.

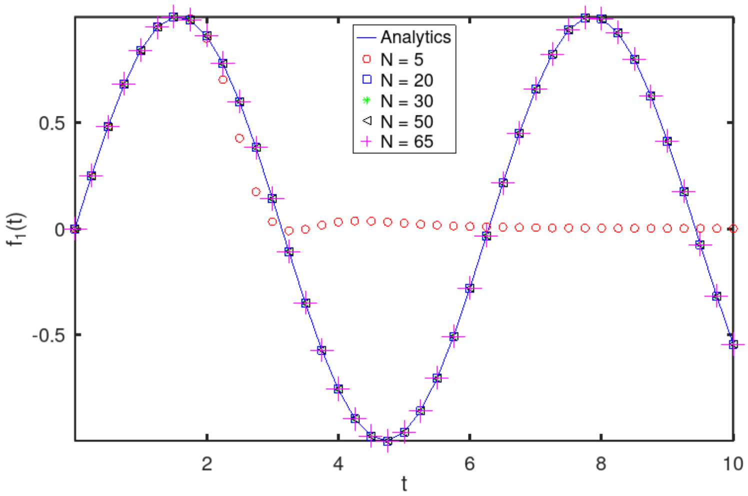

Figure 1: Behavior analysis of Fixed-Talbot Method for $f_1(t)$.

Table 2: Absolute error of the $F_1(s)$ inversion for Fixed-Talbot.

t	$N = 5$	$N = 20$	$N = 30$	$N = 50$	$N = 65$
10^{-4}	6.48436×10^{-10}	1.80248×10^{-18}	1.34441×10^{-17}	7.33190×10^{-14}	3.99379×10^{-12}
1	1.76953×10^{-04}	1.35447×10^{-14}	1.85518×10^{-13}	2.60111×10^{-10}	2.70017×10^{-07}
2	1.21815×10^{-02}	2.76445×10^{-14}	3.94240×10^{-13}	6.16243×10^{-10}	5.40131×10^{-07}
3	1.08590×10^{-01}	2.66731×10^{-14}	4.24577×10^{-13}	5.67706×10^{-10}	9.02584×10^{-07}
4	7.88375×10^{-01}	4.32986×10^{-14}	1.28563×10^{-13}	1.41531×10^{-09}	1.03313×10^{-06}
5	9.84567×10^{-01}	2.74902×10^{-12}	1.29829×10^{-12}	6.92408×10^{-09}	2.23181×10^{-06}
6	2.90080×10^{-01}	5.98130×10^{-10}	1.12521×10^{-13}	1.01512×10^{-09}	1.89346×10^{-06}
7	6.52099×10^{-01}	5.35128×10^{-08}	2.36399×10^{-12}	6.97652×10^{-09}	8.91254×10^{-07}
8	9.86777×10^{-01}	2.14795×10^{-06}	4.39981×10^{-13}	2.49703×10^{-09}	2.18123×10^{-06}
9	4.10581×10^{-01}	3.56365×10^{-05}	2.13118×10^{-12}	2.31501×10^{-09}	1.82522×10^{-06}
10	5.45028×10^{-01}	1.09456×10^{-04}	2.25441×10^{-12}	1.16795×10^{-08}	3.62431×10^{-06}

3.3 The inversion of $F_2(s)$

The Figure 2 and Table 3 present the numerical inversion tests of Laplace Transform through the FT formulation for the function $F_2(s) = \dfrac{1}{(s+1)(s+2)}$. For this function, the FT method provides the best results, using $N = 21$ and with order errors smaller than 10^{-14}, small values of N still allow the adjustment between the numerical profile and the analytical one.

It was observed that the highest values for tested N do not imply obtaining the best results.

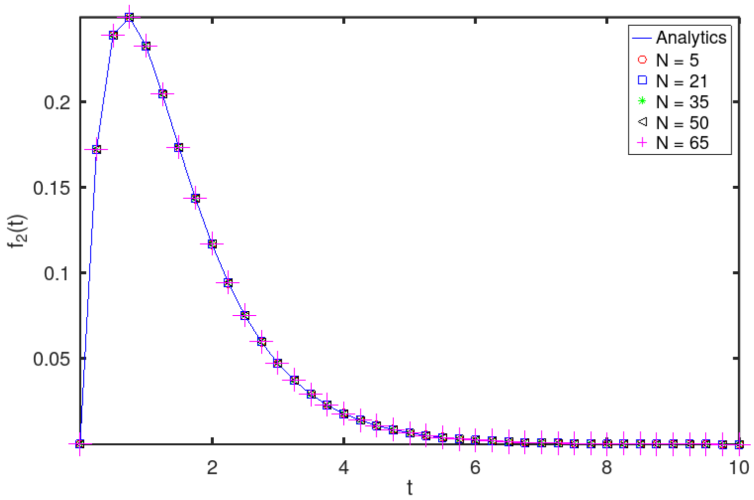

Figure 2: Behavior analysis of Fixed-Talbot Method for $f_2(t)$.

Table 3: Absolute error of the $F_2(s)$ inversion for Fixed-Talbot.

t	$N=5$	$N=21$	$N=35$	$N=50$	$N=65$
10^{-4}	6.48339×10^{-10}	1.14383×10^{-17}	3.97685×10^{-16}	7.55506×10^{-14}	2.22122×10^{-12}
1	5.73609×10^{-05}	2.77555×10^{-16}	1.34264×10^{-12}	2.78010×10^{-10}	2.34412×10^{-07}
2	4.15470×10^{-04}	1.26287×10^{-15}	2.63343×10^{-12}	6.83663×10^{-10}	4.38125×10^{-07}
3	1.35128×10^{-04}	2.49592×10^{-14}	2.85700×10^{-12}	3.87270×10^{-10}	7.00990×10^{-07}
4	2.33827×10^{-04}	3.25781×10^{-15}	5.23285×10^{-12}	8.80894×10^{-10}	5.90035×10^{-07}
5	2.40749×10^{-04}	1.02869×10^{-14}	2.40214×10^{-12}	4.23060×10^{-09}	1.25481×10^{-06}
6	1.11943×10^{-04}	2.83961×10^{-14}	2.70387×10^{-12}	1.99639×10^{-10}	1.15963×10^{-06}
7	8.47163×10^{-06}	2.42423×10^{-14}	2.71073×10^{-12}	3.04613×10^{-09}	5.80466×10^{-07}
8	3.36859×10^{-05}	1.67601×10^{-15}	6.24825×10^{-12}	1.37813×10^{-09}	9.15471×10^{-07}
9	3.67968×10^{-05}	1.41860×10^{-14}	6.01121×10^{-12}	4.51623×10^{-10}	7.10243×10^{-07}
10	2.64633×10^{-05}	3.81195×10^{-15}	3.26354×10^{-12}	4.62844×10^{-09}	1.59920×10^{-06}

3.4 The inversion of $F_3(s)$

Table 4 and Figure 3 show the results obtained using FT method to invert the $F_3(s) = \dfrac{1}{s^2+s+1}$ test function numerically. It was identified that the smallest absolute errors were achieved by FT method using $N = 26$, with error orders lower than 10^{-13}. In the same way as the $F_1(s)$ function, FT method failed to adjust the numerical profile to the analytical one, using small values of N, at least for t from a certain value, that would be bigger than 2.

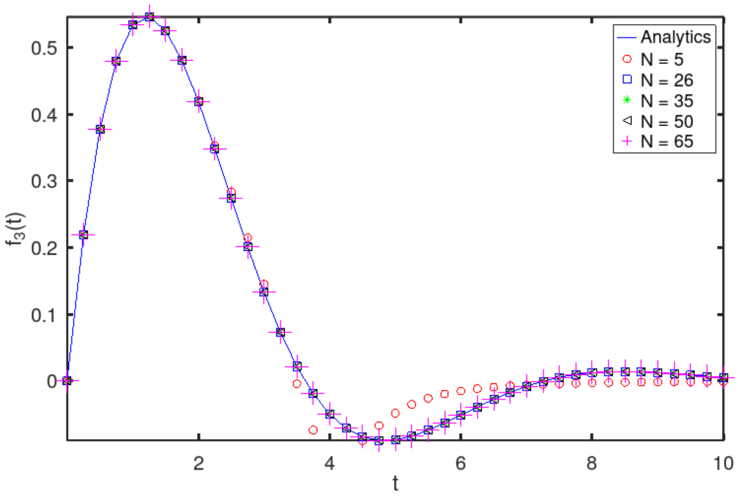

Figure 3: Behavior analysis of Fixed-Talbot Method for $f_3(t)$.

Table 4: Absolute error of the $F_3(s)$ inversion for Fixed-Talbot.

t	$N = 5$	$N = 26$	$N = 35$	$N = 50$	$N = 65$
10^{-4}	6.48404×10^{-10}	3.84891×10^{-18}	3.97712×10^{-16}	7.38701×10^{-14}	5.49314×10^{-12}
1	5.73257×10^{-05}	4.40758×10^{-14}	1.30606×10^{-12}	1.86661×10^{-10}	2.47359×10^{-07}
2	9.19329×10^{-04}	7.09987×10^{-14}	3.86812×10^{-12}	9.08997×10^{-10}	4.46360×10^{-07}
3	1.22272×10^{-02}	1.10467×10^{-14}	2.31509×10^{-12}	5.99913×10^{-10}	8.53194×10^{-07}
4	6.18044×10^{-02}	2.93376×10^{-14}	5.25893×10^{-12}	1.41242×10^{-09}	8.75644×10^{-07}
5	3.98027×10^{-02}	6.63497×10^{-14}	1.67987×10^{-12}	5.85271×10^{-09}	1.89949×10^{-06}
6	3.61843×10^{-02}	1.56541×10^{-13}	3.96650×10^{-12}	7.77158×10^{-10}	1.45922×10^{-06}
7	1.90513×10^{-03}	1.52587×10^{-13}	6.59700×10^{-12}	4.27488×10^{-09}	7.49758×10^{-07}
8	1.53062×10^{-02}	1.38153×10^{-13}	8.95679×10^{-12}	1.89342×10^{-09}	1.34076×10^{-06}
9	1.40374×10^{-02}	2.51534×10^{-15}	8.48610×10^{-12}	1.25930×10^{-09}	8.86057×10^{-07}
10	5.97456×10^{-02}	3.61577×10^{-14}	8.33119×10^{-12}	8.25170×10^{-09}	3.00815×10^{-06}

3.5 The inversion of $F_4(s)$

The results obtained for the numerical inversion tests of Laplace Transform, using FT technique for the function $F_4(s) = -\dfrac{\ln s}{s}$, can be seen in Figure 4 and Table 5. As in the $F_2(s)$ function, the FT method succeeded to adjust the numerical profiles to the analytical ones even for small values of N. The best results were generated by using $N = 20$, with absolute errors of the order of 10^{-12}. For $N > 20$ the FT method precision decreases.

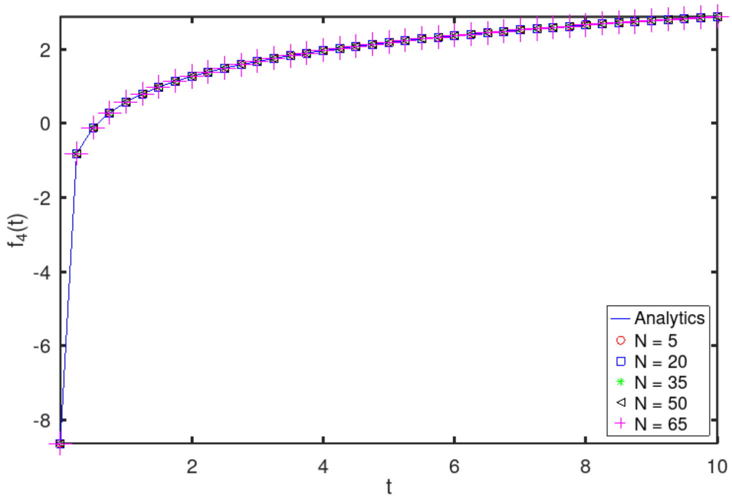

Figure 4: Behavior analysis of Fixed-Talbot Method for $f_4(t)$.

Table 5: Absolute error of the $F_4(s)$ inversion for Fixed-Talbot.

t	$N = 5$	$N = 20$	$N = 35$	$N = 50$	$N = 65$
10^{-4}	5.34802×10^{-04}	1.62359×10^{-12}	5.09967×10^{-10}	1.88693×10^{-07}	4.01118×10^{-05}
1	2.14683×10^{-04}	1.20048×10^{-12}	7.20365×10^{-11}	2.61432×10^{-08}	1.91622×10^{-05}
2	2.71087×10^{-04}	1.14575×10^{-12}	5.33102×10^{-11}	1.67880×10^{-08}	1.44873×10^{-05}
3	3.04082×10^{-04}	1.17150×10^{-12}	3.82551×10^{-11}	6.60517×10^{-09}	1.66946×10^{-05}
4	3.27492×10^{-04}	1.20436×10^{-12}	3.45843×10^{-11}	8.92289×10^{-09}	1.05753×10^{-05}
5	3.45650×10^{-04}	1.28252×10^{-12}	4.97513×10^{-12}	4.55443×10^{-08}	1.38678×10^{-05}
6	3.60486×10^{-04}	1.17683×10^{-12}	1.56488×10^{-11}	3.54277×10^{-09}	9.47657×10^{-06}
7	3.73030×10^{-04}	1.25588×10^{-12}	1.39750×10^{-11}	1.30391×10^{-08}	1.33711×10^{-06}
8	3.83896×10^{-04}	1.21280×10^{-12}	2.31343×10^{-11}	7.01823×10^{-09}	7.42630×10^{-06}
9	3.93481×10^{-04}	1.20747×10^{-12}	2.53352×10^{-12}	4.68760×10^{-09}	4.09336×10^{-07}
10	4.02054×10^{-04}	1.22035×10^{-12}	2.10942×10^{-12}	2.03938×10^{-08}	8.27742×10^{-06}

3.6 The inversion of $F_5(s)$

As shown in Figure 5 and Table 6, the numerical inversion tests of Laplace Transform through the Fixed-Talbot for the function $F_5(s) = \dfrac{1}{(s+1)^2}$ are provided. For this function, the profiles obtained by FT method fitted the analytical one, but the best results were achieved using $N = 21$, with absolute errors no more than 10^{-14}.

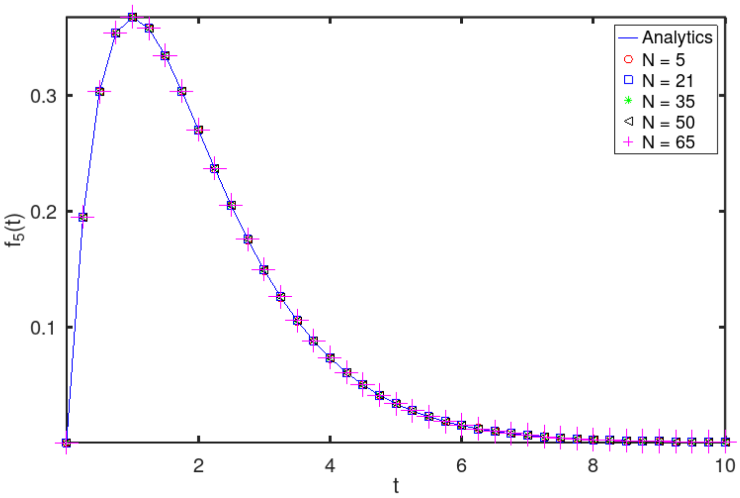

Figure 5: Behavior analysis of Fixed-Talbot Method for $f_5(t)$.

Table 6: Absolute error of the $F_5(s)$ inversion for Fixed-Talbot.

t	$N=5$	$N=21$	$N=35$	$N=50$	$N=65$
10^{-04}	6.48371×10^{-10}	1.13841×10^{-18}	3.64142×10^{-16}	7.11932×10^{-14}	1.99673×10^{-12}
1	4.69460×10^{-05}	3.99680×10^{-15}	1.69370×10^{-12}	3.95322×10^{-10}	2.23726×10^{-07}
2	4.73666×10^{-05}	5.55111×10^{-17}	3.24867×10^{-12}	8.86549×10^{-10}	4.31623×10^{-07}
3	8.57669×10^{-04}	9.32587×10^{-15}	3.24212×10^{-12}	4.32038×10^{-10}	7.54989×10^{-07}
4	5.20617×10^{-04}	1.08246×10^{-14}	4.33092×10^{-12}	1.27362×10^{-09}	8.27356×10^{-07}
5	3.95956×10^{-04}	5.10702×10^{-15}	1.45120×10^{-12}	4.41068×10^{-09}	1.40422×10^{-06}
6	7.15473×10^{-04}	3.94597×10^{-14}	2.92397×10^{-12}	2.78925×10^{-10}	1.21410×10^{-06}
7	4.69021×10^{-04}	2.07629×10^{-14}	8.78986×10^{-13}	3.55986×10^{-09}	7.76610×10^{-07}
8	1.29752×10^{-04}	1.57734×10^{-14}	6.81680×10^{-12}	2.20561×10^{-09}	1.27364×10^{-06}
9	5.73337×10^{-05}	6.45187×10^{-15}	7.05729×10^{-12}	1.10524×10^{-09}	7.02618×10^{-07}
10	1.05502×10^{-04}	1.42407×10^{-14}	3.32815×10^{-12}	6.89744×10^{-09}	2.27664×10^{-06}

3.7 Results Analysis

Based on the absolute mean error (Eq. (7)), it was possible to determine the optimal N on simulation tests for each of the chosen functions, and the corresponding profiles could be included in the Tables 2- 6 and Figures 1-5.

Despite the strong requirement that "reasonable results" would be those whose absolute errors were less than 10^{-8}, the FT method was able to exceed expectations and provide profiles comparable to those obtained through analytical expressions at a very low absolute error.

Through the Figures 1-6 and Tables 2-6, it was possible to observe that: **i)** the best results (with the smallest errors) were obtained for values of N between 20 and 30; **ii)** the use of other values of N (outside this range) were also able to provide so-called "reasonable results" (within the requirement stipulated here), but with greater absolute errors; **iii)** the use of very large N values does not imply refined results; **iv)** in the case of the functions $F_1(s)$ and $F_3(s)$, for small values of N, the FT method was not able to generate profiles adjusted to those obtained by the analytic expressions for the entire period of t; **v)** the functions F_1 and F_3, due to their oscillatory nature, needed more terms in the sum of the approximate integral (Eq. 5) to achieve the best results. In particular, the function F_1 has the slowest convergence according to what was found here; **vi)** the convergence of the functions $F_2(s)$, $F_4(s)$ and $F_5(s)$ is the fastest among all five ones. In particular, $F_2(s)$ and $F_5(s)$ present a very similar convergence rate despite being slightly different; **vii)** from $N=30$, the absolute mean error obtained by the FT method for the five functions seems to grow at a similar rate; **viii)** despite the function $F_4(s)$ being a pseudo-transform associated with a logarithmic function, the numerical inversion by the FT method proved to be faster than those with oscillatory characteristics.

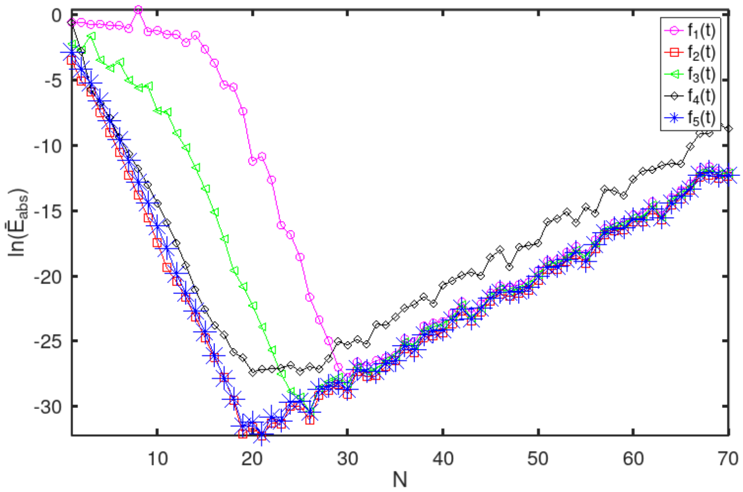

Figure 6: $\overline{E}_{abs}$ behaviour in terms of the parameter N.

4 Conclusions

In this paper, an analysis of Fixed-Talbot method for the Laplace Transform numerical inversion was performed for oscillatory, exponential and logarithmic classes of functions, evaluating the parameter N influence and its efficiency on the treatment of tested functions in a comparative study of the approximate results accuracy in relation to analytical inverse transforms.

The main contribution of this paper is the recommendation of values for the Fixed-Talbot method free parameter N which presents the smallest mean absolute error on the treatment of elementary functions with exponential, oscillatory and logarithmic characteristics.

The use of $\overline{E}_{abs}$ as a tool to identify the best parameter N for inversion of each test function was also important, since it showed that the optimal parameter is an intermediate value, and higher values do not imply better results.

Based on the obtained results, another positive point is the viability of Fixed-Talbot method as an numeric alternative to deal with Laplace Transform inversion, contributing for the expansion of the applicability of the Transform for solving, for example, differential equations.

For a future research, we suggest investigating the application of Fixed-Talbot algorithm to Laplace Transform numerical inversion for functions with different types of behavior, apart from oscillatory, exponential and logarithmic ones.

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